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RULER CONSTRUCTIONS IN CONNEXION WITH CUBIC CURVES.

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VI.

RULER CONSTRUCTIONS IN CONNEXION WITH CUBIC CURVES. By ROBERT RUSSELL, M.A., Fellow and Tutor, Trinity College, Dublin.

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1. The determination of the ninth point common to all plane cubics which pass through eight given points is a fundamental problem in the actual description of the curve when nine points on it are given, and was first solved, by ruler alone, by Sir Andrew Hart in the sixth volume of the "Cambridge and Dublin Mathematical Journal." In the present communication I propose to give a solution of this same problem by an independent method, and to show, by the method adopted, how to determine by ruler only an infinite number of points on the curve.

In a note which will be found at the end of the Paper I give an elementary construction for determining the intersections with the cubic of a line which revolves round a fixed point, and thus complete the construction of the curve.

In what follows, a knowledge of the more elementary conceptions of Sylvester's theory of residuation is assumed, but to save references I shall at once state the two theorems which are here employed :—

If a variable conic intersect a fixed cubic in four fixed points, the line joining the remaining points of intersection passes through a fixed point on the cubic.

A similar property holds for a variable cubic which intersects the fixed cubic in seven given points; and in either case the single resulting point is called the *coresidual* of the given points.

2. In Dr. Salmon's "Higher Plane Curves," Art. 263, it is shown that one, and only one, cubic can be described to pass through eight given points,

so that the point 8 may be the coresidual of the other seven. For, such a cubic being described, is met by any other cubic which passes through the eight given points in a fixed point 9, as yet undetermined, and by the conditions the line 89 meets the cubic again in 8. In other words, the line 89 is a tangent at the point 8 to the required curve, which is therefore unique. I shall now show how to construct the line 89, and the point 9 on it, and thereby the cubic itself.

3. Denoting the point 8 by the co-ordinates x', y', z', three of the given points (1, 2, 3) being taken as vertices of the triangle of reference; if we assume the remaining four (4, 5, 6, 7) to be points common to the conics

$$vy - wz = 0, \quad wz - ux = 0, \quad ux - vy = 0,$$

we can at once determine the lines $u = 0$, $v = 0$, $w = 0$.

We are given five points on the conic $ux - vy = 0$, viz. 4, 5, 6, 7, and the point 3 or $x = 0$, $y = 0$; we can, therefore, by an immediate application of Pascal's Theorem, find the second point in which the line $y = 0$ intersects it. This point is on the line $u = 0$. Similarly, the point in which $z = 0$ intersects $wz - ux = 0$ is in the same line, and therefore the line $u = 0$ is determined.

The construction of this auxiliary triangle, u, v, w, is the central fact in the present discussion.

4. The equation of any cubic which passes through the seven given given points is of the form

$$lx(vy - wz) + my(wz - ux) + nz(ux - vy) = 0,$$

and it is easy to see that the coresidual of these seven points is the point of intersection of the lines

$$\frac{u}{l} = \frac{v}{m} = \frac{w}{n}. \tag{1}$$

If, therefore, the cubic passes through $x'y'z'$, and has that point as the coresidual of the other seven, its equation may be written in either of the equivalent forms

$$\left.\begin{aligned} &u'x(vy - wz) + v'y(wz - ux) + w'z(ux - vy) = 0, \\ \text{or,}\quad &yz(vw' - v'w) + zx(wu' - w'u) + xy(uv' - u'v) = 0. \end{aligned}\right\} \tag{2}$$

The equation of the tangent at 8 may easily be obtained by substituting $\lambda x + \mu x'$, $\lambda y + \mu y$, $\lambda z + \mu z'$ for x, y, z, respectively, and taking that part of the result which is linear, we thus obtain

$$\left.\begin{aligned} & ux'(v'y' - w'z') + vy'(w'z' - u'x') + wz'(u'x' - v'y') = 0, \\ \text{or,} \quad & y'z'(vw' - v'w) + z'x'(wu' - w'u) + x'y'(uv' - u'v') = 0, \end{aligned}\right\} \quad (3)$$

In order to construct this tangent it is only necessary to observe that it passes through the point of intersection of the lines

$$ux' = vy' = w'z,$$

and to determine this point.

Now, the line $ux' - vy' = 0$ meets the conic $ux - vy = 0$ where it meets $xy' - x'y = 0$, and as this latter line passes through one of the five given points on the conic, the second point in which it meets the conic may be found by an application of Pascal's Theorem. Joining this point to the point $u = 0$, $v = 0$, we have the line $ux' - vy' = 0$. Similarly, the line $vy' - wz' = 0$ may be determined, and the line joining their intersection to x', y', z' is the tangent at the point 8, and is a line on which lies the ninth point common to all the cubics.

5. *The Ninth Point.*—From the last paragraph we see that any point on the tangent may be defined by the equations

$$u = \lambda u' + \frac{\mu}{x'}, \quad v = \lambda v' + \frac{\mu}{y'}, \quad w = \lambda w' + \frac{\mu}{z'}.$$

Substituting these values for u, v, w in (2), we conclude that the required point 9 lies on the conic whose equation is

$$yzx'(v'y' - w'z') + zxy'(w'z' - u'x') + xyz'(u'x' - v'y') = 0. \quad (4)$$

This conic passes the points 1, 2, 3, 8 and the point of intersection of the lines $xu' = yv' = zw'$. The position of this last point, which also lies on the cubic, may be determined by a process similar to that adopted in Art. 4 for obtaining the point $ux' = vy' = wz'$. The tangent to the cubic at the the point 8 meets this conic again in the point 9, which can, therefore, be found by means of Pascal's Theorem.

It may be observed that the only instrument required in the above is a ruler; for the only geometrical construction involved is that of determining the line joining the intersections of opposite sides of a Pascal hexagon.

6. I shall now indicate, as briefly as possible, the method by which an indefinitely great number of points on the cubic may be determined.

The cubic is met by the line $x = 0$ in the two given points 2, 3, and by the conic $vy - wz = 0$ in the points 1, 4, 5, 6, 7, and they each meet it again in one other point. Both of these points lie on the line $vw' - v'w = 0$, and may, therefore, at once be found—the former as a direct intersection, and the latter by the usual application of Pascal's Theorem.

Denoting the sixth point of intersection of the above conic with the cubic by the symbol 1′ and by 2′, 3′, the corresponding points related to the other conics, we easily see that these points lie, respectively, on the on the lines

$$yv' - zw' = 0, \quad zw' - xu' = 0, \quad xu' - yv' = 0,$$

and that 1, 1′; 2, 2′; 3, 3′ connect through the point $xu' = yv' = zw'$ (see Art. 5), and therefore this point, which we shall denote by K, is the coresidual of the points 4, 5, 6, 7.

7. We can now make an important deduction. If the seven given points had been 1′, 2′, 3′, 4, 5, 6, 7, we should have the same triangle uvw, and

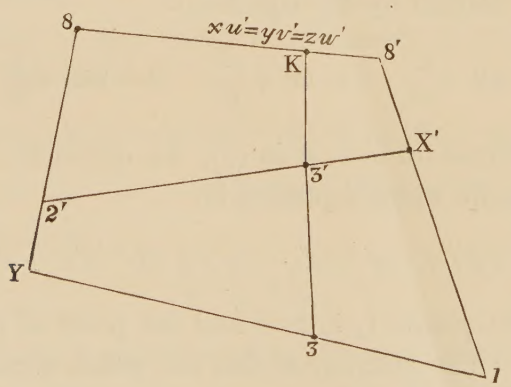


Fig. 1.

the same system of conics, and if we can determine that point on the cubic which is the coresidual of the above seven, we shall have a second source for obtaining additional points on the curve. To obtain this point, the

conic through 1′, 4, 5, 6, 7, and the line joining 2′, 3′, constitute a particular cubic through the seven points whose remaining points of intersection with the cubic are 1 and X', and we have to find where $1X'$ meets the cubic again. Denote this point by 8′. We saw, in Art. 6, that 13 and 82′ meet on the cubic, and that 33′ passes through the point K; therefore the line joining 8 to K meets the cubic in the required point (see fig. 1).

There are several ways of determining 8′, of which the following is one :—The co-ordinates of any point on the line 88′ may be written

$$x = \lambda x' + \frac{\mu}{u'}; \quad y = \lambda y' + \frac{\mu}{v'}; \quad z = \lambda z' + \frac{\mu}{w'}.$$

Replacing $u'x$, $v'y$, $w'z$ by $\lambda u'x' + \mu$, $\lambda v'y' + \mu$, $\lambda w'z' + \mu$ in the equation of the cubic, we obtain

$$u'x'\,(vy - wz) + v'y'\,(wz - ux) + w'z'\,(ux - vy) = 0,$$

a conic on which 8′ lies, and which passes through the five known points 8, 4, 5, 6, 7. Pascal's Theorem determines the position of 8′.

8. *The Polar Conics of* 8, 8′, *and* K.—If from a point P on a cubic a variable line be drawn across it, the locus of the harmonic conjugate of P with respect to the other two points in which the cubic meets the line is the the polar conic of P. The polar conic of 8, therefore, passes through 8 and its harmonic conjugates with respect to 1′, X; 2′, Y; 3′, Z, and 8′, K; and likewise the polar conic of 8′ passes through 8′ and its harmonic conjugates with respect to 1, X'; 2, Y'; 3, Z', and 8, K. We are thus given five points on each. The polar conic of K may similarly be determined, for it passes through that point and its harmonic conjugates with respect to 1, 1′; 2, 2′; 3, 3′; 8, 8′; and, similarly, the polar conic of any point on the cubic can be determined if we know the complete intersections with the cubic of any four lines which pass through it. We can thus at will obtain, at least, five points each on the polar conics of 1, 2, 3, X, Y, Z, &c., . . .

9. We are now in a position to construct an infinite number of points on the curve; for the line joining any known point P to any of the points whose polar conics have been determined, meets that conic in a point Q, found by Pascal's Theorem, and the remaining point on the line P 1, say, is the harmonic conjugate of P with respect to Q 1.

It is obvious that the method admits of infinite repetition.

10. *Cubic through Nine given Points.*—I conclude by showing that if we are given nine points on a cubic we can, by ruler, find that point on it which is the coresidual of any seven that may be selected, and thus reduce the problem of determining the curve to that which we been discussing. Denote the seven selected by the numbers 1, 2, 3, 4, 5, 6, 7, and, as before, take 1, 2, 3 as vertices of triangle of reference, and determine $u = 0$, $v = 0$, $w = 0$, as in Art. 3.

The equation of the cubic passing through the nine points is

$$\xi x(vy - wz) + \eta y(wz - ux) + \zeta z(ux - vy) = 0,$$

where ξ, η, ζ are constants determined by substituting, in this equation, the co-ordinates x', y', z'; x'', y'', z'' of the eighth and ninth points. The equations we thus get for determining ξ, η, ζ are

$$\xi x'(v'y' - w'z') + \eta y'(w'z' - u'x') + \zeta z'(u'x' - v'y') = 0,$$

$$\xi x''(v''y'' - w''z'') + \eta y''(w''z'' - u''x'') + \zeta z''(u''x'' - v''y'') = 0,$$

which are of the same form as the equations of the tangents at the points 8, 9 to the coresidual cubics passing through 1, 2, 3, 4, 5, 6, 7, 8, and 1, 2, 3, 4, 5, 6, 7, 9. It has been shown, in Art. 4, how these tangents are constructed, and their intersection is that point on the required cubic which is the coresidual of the seven selected.

NOTE.

CONTINUOUS DESCRIPTION OF THE CUBIC.

In Art. 10 the solution of the problem has been reduced to that of drawing a cubic through seven of the points which shall have a determined point $x'y'z'$ ($\xi\eta\zeta$ of Art. 10) as the coresidual of these seven. The equation of the line joining $x'y'z'$ to any other point $x''y''z''$, referred to the angle uvw, is

$$u''(vw' - v'w) + v''(wu' - w'u) + w''(uv' - u'v) = 0, \qquad (6)$$

and this, combined with the equation of the curve

$$yz(vw' - v'w) + zx(wu' - w'u) + xy(uv' - u'v) = 0,$$

gives for the determination of the other two points on this line

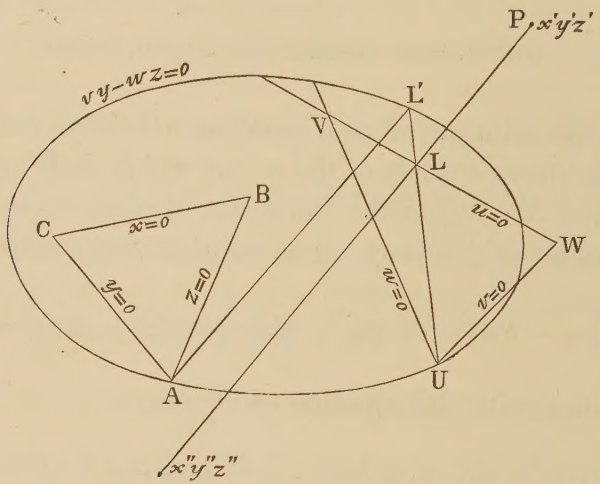
$$\frac{vw' - v'w}{x(yv'' - zw'')} = \frac{wu' - w'u}{y(zw'' - xu'')} = \frac{uv' - u'v}{z(xu'' - yv'')},$$

from which which we deduce

$$\left.\begin{aligned} &u'x(yv'' - zw'') + v'y(zw'' - xu'') + w'z(xu'' - yv'') = 0, \\ \text{or,}\quad &yz(v'w'' - v''w') + zx(w'u'' - w''u') + xy(u'v'' - u''v') = 0, \end{aligned}\right\} \qquad (7)$$

the equation of a conic passing through four fixed points, viz. 1, 2, 3 and K. To fix the conic we must obtain either another point or a tangent at some one of these four. The tangent at

$$1 \text{ or } y = 0,\ z = 0, \text{ is } y(u'v'' - u''v') + z(w'u'' - w''u') = 0,$$

and may be constructed as follows: Let the variable line represented by

equation (6) intersect the line $u = 0$ in the point L; join UL, and produce it to meet the conic $vy - wz = 0$ in L', then the line AL' is the tangent at A to the conic which passes through 1, 2, 3, K, and whose intersections with LP describe the cubic.